Discontinuous Functions

Yicong Lin

November, 2019

Continuity of a function, say f, is defined by using the concept of limits. To understand discontinuity, we shall first discuss the situations in which limits do not exist. Before starting, let's first recall a concept which is confused to most of the students.

0.1 Divergence to ∞

We say f diverges to ∞, while $x \to x_0$ or $x \to \infty$, it should be clear that given an *arbitrarily large* number $H \in \mathbb{R}$, we should be able to find a neighborhood of x_0 or ∞ such that f never return below H. This feature distinguishes divergence to ∞ from unboundedness. As I mentioned in the tutorial, if f diverges to ∞, while $x \to x_0$ or $x \to \infty$, it must be unbounded. However, if f is unbounded, it is not necessarily divergent to ∞.

Example. 0.1 *The sequence* $\{1, 2, 1, 4, 1, 6, 1, 8, \cdots\}$ *is unbounded but is not divergent to* ∞ *because it always returns to* 1.

Example. 0.2 *Let* f *be constructed as* $f(x) = \frac{1}{x-1}\sin\left(\frac{1}{x-1}\right)$, $x \in (0, 1)$. *Consider the behavior of* f *when* x *approaches to* $x = 1$. *The graphs are given as below*

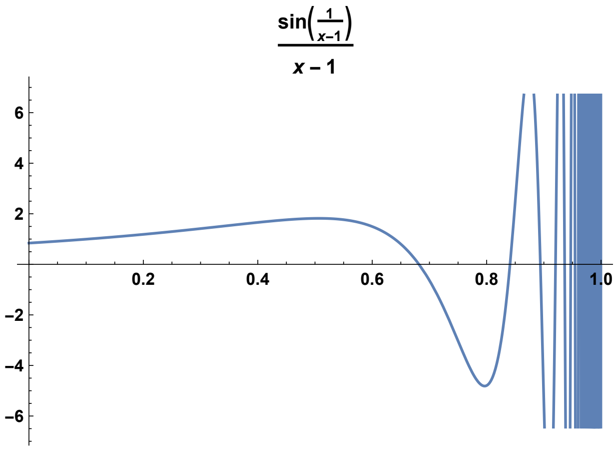


Figure 1: x is taken in $(0, 1)$

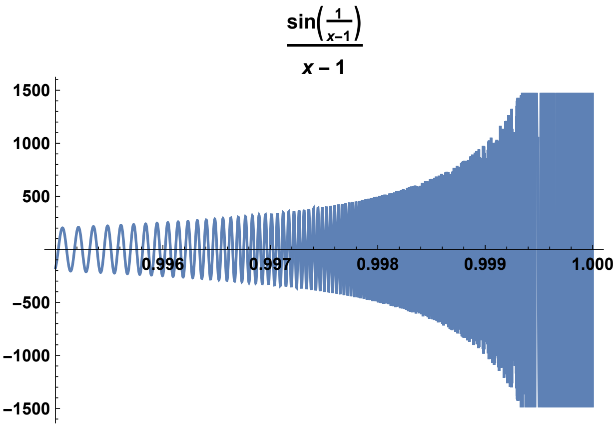


Figure 2: Let's zoom in and see $x \in (0.995, 1)$.

It is clearly seen that when $x \to 1$, the function is oscillating around the horizontal line, in the meanwhile, getting explosive (see the values of y-axis). Apparently, this function is unbounded (explosive), it is not divergent to ∞ no matter how close when x approaches 1.

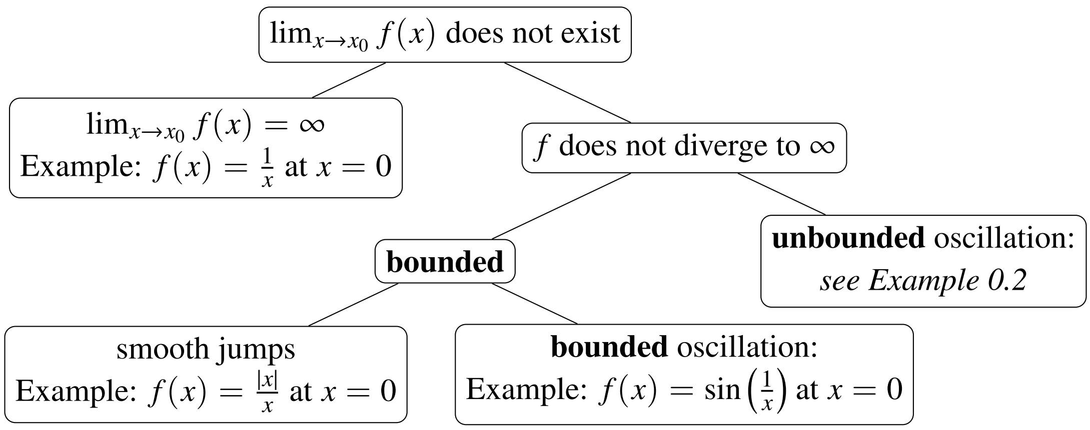
In summary, if there exists a point $x_0 \in \mathbb{R}$,

$$\left\{ f : \lim_{x \to x_0} f(x) = \infty \text{ or } \lim_{x \to \infty} f(x) = \infty \right\} \subset \{ f : f \text{ is unbounded} \} . \tag{1}$$

Equivalently, we can say if f does not diverge to ∞ at any $x = x_0 \in \mathbb{R}$ or while $x \to \infty$, f can be either bounded or unbounded.

0.2 Nonexistence of Limits

We move on to investigate the cases in which limits do not exist. To my best of knowledge, they can be summarized as follows.



0.3 Discontinuity

Finally, we can summarize the cases of discontinuities as follows.