A Collection of Propositions in MWG

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Chapter 2

Proposition 2.E.1: If the Walrasian demand function $x(p,w)$ is homogeneous of degree zero, then for all p and w:

$$\sum_{k=1}^{n} \frac{\partial x_\ell(p,w)}{\partial p_k} p_k + \frac{\partial x_\ell(p,w)}{\partial w} = 0 \quad for\ \ell = 1, ..., L.$$

In matrix notation, this is expressed as

$$D_p x(p,w)\, p^T + D_w x(p,w)\, w = 0$$

Using elasticities, condition takes the following form:

$$\sum_{k=1}^{L} \varepsilon_{\ell k}(p,w) + \varepsilon_{\ell w}(p,w) = 0 \quad for\ \ell = 1, ..., L.$$

Proposition 2.E.2: If the Walrasian demand function $x(p,w)$ satisfies Walras' law, then for all p and w:

$$\sum_{\ell=1}^{L} p_l \frac{\partial x_\ell(p,w)}{\partial p_k} + x_k(p,w) = 0 \quad for\ k = 1, ..., L,$$

or, written in matrix notion,

$$pD_p(x,w) + x(p,w)^T = 0$$

Proposition 2.E.3 : If the Walrasian demand function $x(p,w)$ satisfies Walras' law, then for all p and w:

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(p,w)}{\partial w} = 1$$

or, written in matrix notion,

$$pD_w(x,w) = 1$$

Proposition 2.F.1 : Suppose that the Walrasian demand function $x(p,w)$ is homogeneous of degree zero and satisfies Walras' law. Then $x(p,w)$ satisfies the weak axiom if and only if the following property holds:

For any compensated price change from an initial situation (p,w) to a new price wealth pair $(p',w') = (p', p' - x(p,w))$, we have

$$(p' - p)[x(p',w') - x(p,w)] \leqslant 0$$

with strict inequality whenever $x(p,w) \neq x(p',w')$.

Proposition 2.F.2 : If a differentiable Walrasian demand function $x(p,w)$ satisfies Walras' law,

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homogeneity of degree zero, and the weak axiom, then at any (p, w), the Slutsky matrix S(p,w) satisfies $vS(p,w)v^T \leqslant 0$ for any $v \in \mathbf{R}^L$.

Proposition 2.F.3 : Suppose that the Walrasian demand function $x(p,w)$ is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then $pS(p,w) = 0$ and $S(p,w)p^T = 0$ for any (p,w).

Chapter 3

Proposition 3.C.1 : Suppose that the rational preference relation $\succsim$ on X is continuous. Then there is a **continuous** utility function u(x) that $\succsim$. (I guess it is not enough, the monotonicity should not be omitted.)

Proposition 3.D.1 : If $p \gg 0$ and u(·) is continuous, then the utility maximization problem has a solution.

Proposition 3.D.2 : Suppose that u(·) is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbf{R}^L_+$. Then the Walrasian demand correspondence x(p, w) possesses the following properties:

(i) Homogeneity of degree zero in (p,w): $x(\alpha p, \alpha w) = x(p,w)$ for any p, w and scalar $\alpha > 0$.

(ii) Walras' raw: $p \cdot x = w$ for all $x \in x(p,w)$. (No excess money)

(iii) Convexity/uniqueness: If $\succsim$ is convex, so that u(·) is **quasiconcave**, then $x(p,w)$ is a convex set. Moreover, if $\succsim$ is strictly convex, so that u(·) is strictly quasiconcave, then $x = (p,w)$ consists of a singe element.

Proposition 3.D.3 : Suppose that u(·) is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbf{R}^L_+$. The indirect utility function $v(p,w)$ is

(i) Homogeneous of degree **zero**.

(ii) Strictly **increasing** in w and non**in**creasing in p_ℓ for any ℓ.

(iii) **Quasiconvex**; that is, the set $\{(p,w) : v(p,w) \leqslant \bar{v}\}$ is convex for any $\bar{v}$.

(iv) Continuous in p and w.

Note that the direct utility function is independent of prices and income, whereas the indirect utility function is independent of quantities of goods. An indifference curve corresponding to the indirect utility function is convex but represents a higher preference if it is closer to the origin (The shape is like direct utility function but the value is higher if it is closer to origin). Therefore, **optimizing with the direct utility function is a problem of maximizing under the assumption of given prices and income, whereas optimizing with the indirect utility function is a problem of minimizing (usually prices) under the assumption of given quantities.**

Proposition 3.E.1 : Suppose that u(·) is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbf{R}^L_+$ and that the price vector is $p \gg 0$. We have

(i) If x^* is optimal in the UMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the minimized expenditure level in this EMP is exactly w.

(ii) If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximized utility level in this UMP is exactly u.

Proposition 3.E.2 : Suppose that u(·) is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbf{R}^L_+$. The expenditure function $e(p,u)$ is

(i) Homogeneous of degree **one** in p.

(ii) Strictly **increasing** in u and non**de**creasing in p_ℓ for any ℓ.

(iii) **Concave** in p.

(iv) Continuous in p and u.

Proposition 3.E.3 : Suppose that u($\cdot$) is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbf{R}^L_+$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ possesses the following properties:

(i) Homogeneity of degree **zero** in p: $h(\alpha p, u) = h(p, u)$ for any p, u and $\alpha > 0$.

(ii) No excess utility: For any $x \in h(p, u), u(x) = u$.

(iii) Convexity/uniqueness: If $\succsim$ is convex, then $h(p, u)$ is a **convex** set, and $\succsim$ is strictly convex, so that u($\cdot$) is strictly quasiconcave, then there is a unique element in $h(p, u)$.

Proposition 3.E.4 : Suppose that u($\cdot$) is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ and that $h(p, u)$ consists of a single element for all $p \gg 0$. Then the Hicksian demand function $h(p, u)$ satisfies the compensated law of demand: For all p' and p'',

$$(p'' - p') \cdot [h(p'', u) - h(p', u)] \leqslant 0$$

Proposition 3.F.1 : (The Duality Theorem). Let K be a nonempty closed set, and let $\mu_K(\cdot)$ be its support function. Then there is a unique $\bar{x} \in K$ such that $\bar{p} \cdot \bar{x} = \mu_K(\bar{p})$ if and only if $\mu_K(\cdot)$ is differentiable at $\bar{p}$. Moreover, in this case,

$$\mu_K(\bar{p}) = \bar{x}$$

Proposition 3.G.1 : Suppose that u($\cdot$) is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbf{R}^L_+$. For all p and u, the Hicksian demand $h(p, u)$ is the derivative vector of the expenditure function with respect to prices:

$$h(p, u) = \nabla_p e(p, u).$$

That is, $h_\ell(p, u) = \frac{\partial e(p,u)}{\partial p_\ell}$ for all $\ell = 1, ..., L$.

Proposition 3.G.2 : Suppose that u($\cdot$) is a continuous utility function representing a locally nonsatiated and **strictly convex** preference relation $\succsim$ defined on the consumption set $X = \mathbf{R}^L_+$. Suppose also that $h(\cdot, u)$ is continuously differentiable at (p, u), and denote its $L \times L$ derivative matrix by $D_p h(p, u)$. Then

(i) $D_p h(p, u) = D_p^2 e(p, u)$.

(ii) $D_p h(p, u)$ is a negative semidefinite matrix.

(iii) $D_p h(p, u)$ is a **symmetric** matrix.

(iv) $D_p h(p, u) p^T = 0$.

Proposition 3.G.3 : (The Slutsky Equation) Suppose that u($\cdot$) is a continuous utility function representing a locally nonsatiated and **strictly convex** preference relation $\succsim$ defined on the consumption set $X = \mathbf{R}^L_+$. Then for all (p, w), and $u = v(p, w)$, we have

$$\frac{\partial h_\ell(p, v)}{\partial p_k} = \frac{\partial x_\ell(p, w)}{\partial p_k} + \frac{\partial x_\ell(p, w)}{\partial w} x_k(p, w)$$

for all ℓ, k, or equivalently, in matrix notation,

$$D_p h(p, u) = D_p x(p, w) + D_w x(p, w) x(p, w)^T.$$

By Proposition 3.G.2 & 3.G.3, when demand is generated from preference maximization, $S(p, w)$ must be negative semidefinite **symmetric**, and satisfy $S(p, w)p^T = 0$.

Proposition 3.G.4 : (Roy's Identity) Suppose that u($\cdot$) is a continuous utility function representing a locally nonsatiated and **strictly convex** preference relation $\succsim$ defined on the consumption set $X = \mathbf{R}^L_+$. Suppose also that the indirect utility function is differentiable at $(\bar{p}, \bar{w}) \gg 0$. Then

$$x(\bar{p}, \bar{w}) = -\frac{1}{\nabla_w v(\bar{p}, \bar{w})} \nabla_p v(\bar{p}, \bar{w}).$$

That is, for every $\ell = 1, ..., L$:

$$x_\ell(\bar{p}, \bar{w}) = -\frac{\partial v(\bar{p}, \bar{w})/\partial p_\ell}{\partial v(\bar{p}, \bar{w})/\partial w}$$

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